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LETTER TO THE EDITOR

Finite size studies of the Ashkin–Teller model

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Abstract. We study the Ashkin–Teller model in the time-continuous Hamiltonian version. Finite-size scaling is used to calculate the magnetic (γ_M), electric (γ_P) and thermal (α and ν) critical exponents for several values of the coupling constant (λ). Our results confirm the believed extended scaling relations and suggest a conjecture relating the mass-gap amplitudes and critical indices in the Hamiltonian context.

Since the work of Wegner (1972), showing that the two-dimensional (2D) Ashkin–Teller (AT) model is a staggered eight-vertex model, several studies have been done concerning its critical behaviour. In particular, some relations between the critical exponents, proposed by Enting (1975) and Kadanoff (1979), have been derived by exploring the relationship of the eight-vertex model to the generalised Villain (1975), model (Kadanoff 1979, Kadanoff and Brown 1979). These extended scaling relations are

$$x_\varepsilon^{AT} = 1/x_\varepsilon^{8V}, \qquad x_M^{AT} = 1/8, \qquad x_P^{AT} = x_\varepsilon^{AT}/4, \tag{1a, b, c}$$

where $x_\varepsilon^{AT}(x_\varepsilon^{8V})$ is the correlation function exponent of the energy (density) for the Ashkin–Teller (eight-vertex) model and $x_M^{AT}(x_P^{AT})$ that of magnetisation (polarisation). According to these relations the usual critical indices are given by

$$1/\nu = 2 - \pi/2 \cos^{-1}[\tanh(2K_4)/(\tanh(2K_4) - 1)], \tag{2a}$$

$$\alpha/\nu = 2/\nu - 2, \tag{2b}$$

$$\gamma_M/\nu = 7/4, \tag{2c}$$

$$\gamma_P/\nu = 1 + 1/2\nu, \tag{2d}$$

where K_4 is the four-spin coupling constant of the model.

More recently a 1D quantum Hamiltonian analogue of the 2D AT model was introduced and investigated by Kohmoto *et al* (1981). Their results corroborate the validity of the extended scaling relations mentioned above, with the translation

$$\tanh(2K_4)/[\tanh(2K_4) - 1] \to -\lambda, \tag{3}$$

λ being a Hamiltonian coupling constant (see equation (4)).

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In this letter we present a finite-size scaling (FSS) study of that quantum (time-continuous) Hamiltonian†

$$H=\sum_i \{(1-\sigma^x(i))+(1-\tau^x(i))+\lambda(1-\sigma^x(i)\tau^x(i))$$
$$-\beta[\sigma^z(i)\sigma^z(i+1)+\tau^z(i)\tau^z(i+1)+\lambda\sigma^z(i)\sigma^z(i+1)\tau^z(i)\tau^z(i+1)]\}, \quad (4)$$

in the region $-\frac{1}{2}\leq\lambda\leq 1$ where, as shown in figure 1, the model is expected to exhibit a single phase transition at its self-dual point $\beta=1$. In equation (4) $\sigma^x(i)$, $\sigma^z(i)$, $\tau^x(i)$, $\tau^z(i)$ are two sets of Pauli matrices associated with site i and the parameter β plays the role of inverse temperature.

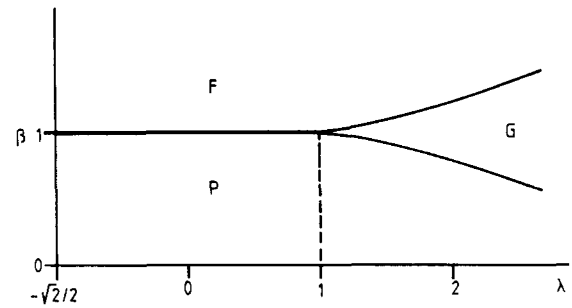

Figure 1. The expected phase diagram for the AT Hamiltonian. The diagram exhibits three phases: the ferromagnetic ordered phase F, the partially ordered G and the disordered paramagnetic phase P. All the critical lines are related to continuous phase transitions. In this letter we are interested in the section of critical line denoted by a bold line which contains, as particular cases, the doubled Ising ($\lambda=0$) and the four-state Potts model ($\lambda=1$) critical points.

The fundamental assumption of the FSS theory (Barber 1983) is that the mass gap G (related to the correlation length), which in an infinite system varies near the critical coupling‡ β_c as

$$G_\infty(\beta)=E_1-E_0\sim(\beta-\beta_c)^\nu, \quad (5)$$

for a finite system of size L, behaves as

$$G_L(\beta)|_{\beta=\beta_c}\sim L^{-1}. \quad (6)$$

In Equation (5) $E_0(E_1)$ is the energy of the ground (first-excited)-state of the Hamiltonian H. In general, any thermodynamical quantity $T(\beta)$ whose behaviour in the infinite lattice system is

$$T_\infty(\beta)\sim(\beta-\beta_c)^{-\psi}, \quad (7)$$

† Another time-continuous Hamiltonian was studied by Drugowich de Felício and Köberle (1982), its FSS being currently under investigation.

‡ In practice the critical coupling β_c of the infinite system is often unknown. The finite-size scaling form (6) suggests that β_c can be found from the sequence of values β for which successive ratios of $G(\beta, L)$ and $G(\beta, L-1)$ exactly scale, i.e., the value of β for which

$$R_L(\beta)=LG(\beta, L)/(L-1)G(\beta, L-1)=1.$$

in the finite system it will be

$$T_L(\beta)|_{\beta=\beta_c} \sim L^{\psi/\nu}. \tag{8}$$

Therefore, by considering a set of finite lattices it is possible to estimate the index ψ/ν by extrapolating the sequence

$$L[T_L(\beta_c) - T_{L-1}(\beta_c)]/T_{L-1}(\beta_c) \to \psi/\nu. \tag{9}$$

In order to obtain the critical exponents ν, α/ν, γ_M/ν and γ_P/ν it is convenient to perform the extrapolations with the following functions: the β-derivative of the mass-gap $(\partial G/\partial\beta|_{\beta=\beta c})$, the 'specific heat' $(\partial^2 E_0/\partial\beta^2|_{\beta=\beta_c})$, the magnetic susceptibility $(\partial^2 E_0/\partial h^2|h=0)$ and the electrical susceptibility $(\partial^2 E_0/\partial\xi^2|_{\xi=0})$ respectively.

Our results for the critical indices are summarised in tables 1 and 2. It is worthwhile to mention that the high precision achieved is due to the use of Vanden Broeck–Schwartz approximants (Vanden Broeck and Schwartz 1979, Hamer and Barber 1981). The poor convergence in the $\lambda \simeq 1$ region is to be expected since the case $\lambda = 1$ corresponds to the four-state Potts model in which marginality effects are important.

The remarkable agreement of our results with equations (2) and (3) strongly supports the extended scaling relations in the Hamiltonian formulation.

Furthermore we wish to point out that, in connection with this analysis, we have found an important property relating the mass-gap amplitudes and critical exponents. In order to state this property we remind the reader that under a continuous phase

Table 1. Estimated (FSS) and conjectured results for the thermal (ν and α) critical exponents of the 2D Ashkin–Teller model. The parameter λ is related to the four-spin coupling constant (see equation (3)).

λ	$1/\nu$ (FSS)	$1/\nu$ (conjectured)	α/ν (FSS)	α/ν (conjectured)
−0.50	0.51 ± 0.02	0.5000	−0.98 ± 0.05	−1.0000
−0.25	0.808 ± 0.001	0.8083	−0.39 ± 0.02	−0.3833
0.10	1.0599 ± 0.0001	1.0599	0.19 ± 0.02	0.1198
0.25	1.138 ± 0.001	1.1385	0.27 ± 0.03	0.2771
0.35	1.185 ± 0.001	1.1854	0.38 ± 0.02	0.3708
0.50	1.250 ± 0.001	1.2500	0.504 ± 0.005	0.5000
0.75	1.348 ± 0.003	1.3506	0.685 ± 0.005	0.7012
0.85	1.37 ± 0.03	1.3927	0.74 ± 0.01	0.7855
1.00	1.42 ± 0.05	1.5000	0.77 ± 0.02	1.0000

Table 2. Estimated (FSS) and conjectured results, for several values of coupling λ, for the magnetic and electric susceptibilities (γ_M and γ_P).

λ	γ_P/ν (FSS)	γ_P/ν (conjectured)	γ_M/ν (FSS)	γ_M/ν (conjectured)
−0.50	1.253 ± 0.005	1.25000	1.73 ± 0.02	1.75000
−0.25	1.405 ± 0.001	1.40415	1.73 ± 0.03	1.75000
0.25	1.5692 ± 0.0002	1.56929	1.750 ± 0.001	1.75000
0.50	1.625 ± 0.001	1.62500	1.750 ± 0.001	1.75000
0.75	1.675 ± 0.001	1.67530	1.751 ± 0.002	1.75000
0.85	1.696 ± 0.001	1.69638	1.752 ± 0.002	1.75000
1.00	1.743 ± 0.005	1.75000	1.743 ± 0.005	1.75000

transition (in a truly infinite system) a whole set of eigenstates degenerates with the ground state. Consequently we can define several mass gaps G_i relating the energy of those states E_i to the ground state one E_0. Under the fundamental assumption of FSS theory those mass gaps for a finite system of size L should behave as

$$G_L^i(\beta)|_{\beta=\beta_c} = A_i/L$$

where the corresponding amplitude was denoted by A_i. These mass gaps are related to the long distance behaviour of different correlation functions.

In this basis in which σ^x and τ^x are diagonal the parity operator $\pi\sigma^x(i)\tau^x(i)$ is also diagonal. Using this basis it is not difficult to convince oneself that while the first and second mass gap are related to spin–spin correlation functions $\langle\sigma^z(j)\sigma^z(j+n)\rangle$ and $\langle\sigma^z(j)\tau^z(j)\sigma^z(j+n)\tau^z(j+n)\rangle$ the third relevant gap is related to the energy–energy correlation function. We have calculated those three amplitudes for finite lattices and extrapolated them, via Padé approximants, for the infinite lattice. Our results are summarised in table 3. Those numbers together with equations (1) and (2) suggest the existence of a relation between amplitudes and critical indices in the Hamiltonian finite-size scaling, namely

$$A_i = 4\pi\beta_c x_i/x_\varepsilon \qquad (10)$$

where $x_i(x_\varepsilon)$ is the anomalous dimension of the operator related to the mass gap G_i (energy operator) and, as before, β_c is the critical coupling. In the AT model x_1, x_2 and x_3 are the anomalous dimensions of the magnetisation, polarisation and energy operator respectively. Our claim is that the above relation should be a general property of lattice quantum Hamiltonians†.

Table 3. Mass-gap amplitudes (A_i) in units of ($4\pi\chi_i\beta_c/\chi_\varepsilon$). We remember that in our case $\beta_c = 1$, χ_1 is the anomalous dimension of the magnetisation operator, χ_2 is the anomalous dimension of the polarisation operator and $\chi_3 = \chi_\varepsilon$ that of the energy operator.

λ	A_1	A_2	A_3
−0.50	0.947 43	0.974 28	0.974 36
−0.25	0.993 44	0.993 86	0.994 13
0.10	0.999 05	0.999 05	0.999 08
0.25	0.993 98	0.993 96	1.004 74
0.35	0.988 02	0.987 94	0.996 99
0.50	0.974 70	0.974 27	1.004 22
0.75	0.930 58	0.933 68	1.031 03
0.85	0.895 16	0.905 54	1.051 54
1.00	0.755 36	0.755 36	1.096 31

The above conjecture is the Hamiltonian counterpart of another one recently introduced (Luck 1982, Derrida and de Seze 1982, Nightingale and Blöte 1983) in the transfer matrix context, and it is probably a consequence of conformal invariance of the underlying field theory in the vicinity of the critical coupling (Cardy 1984).

Finally we want to comment on the numerical part of calculations. In this analysis we have considered lattices up to $L = 9$ sites (4 states per site) and evaluated the lowest

† We are checking this property in the Baxter model whose Hamiltonian can be described in a form similar to that of equation (4) (Líbero and Drugowich de Felício 1983).

eigenvalues of the Hamiltonian H within the machine precision (10^{-15}) using the Lanczos scheme of tridiagonalisation (Whitehead *et al* 1977, Roomany *et al* 1980). In order to save computer memory we represent any quantum state by a pair of integer numbers whose binary code gives its spin configuration. Because we are dealing with periodic boundary conditions we keep only one pair of numbers to represent a whole family of cyclic invariant states. The action of any quantum operator on a given state can be implemented by logical functions usually built in advanced computer languages. In addition, to minimise searching and storage time we employ hash-tables (Knudth 1973, Alcaraz and Drugowich de Felício 1984).

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